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THE DIAMETER OF δ-PINCHED MANIFOLDS

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0. Introduction

It is interesting to investigate the manifold structures of a complete riemannian manifold whose sectional curvature is bounded below by a positive constant. As is well known such a riemannian manifold is compact and we may suppose that its sectional curvature K_σ satisfies $0 < \delta \leq K_\sigma \leq 1$ for every plane section σ. Berger proved in [2] and [3] that a complete, simply connected and even dimensional riemannian manifold with $\delta = 1/4$ is homeomorphic to a sphere, or otherwise M is isometric to one of the compact symmetric spaces of rank one. For arbitrary dimensional riemannian manifolds, Klingenberg proved in [8] that a complete and simply connected riemannian manifold with $\delta > 1/4$ is homeomorphic to a sphere. Moreover, Berger claimed in [4] that M is a homology sphere if the diameter $d(M)$ of M satisfies $d(M) > \pi/(2\sqrt{\delta})$ for $0 < \delta \leq 1$.

Since the diameter $d(M)$ of a δ-pinched manifold M plays an important role in the proofs of these interesting results mentioned above, it might be significant to investigate the relationship between the manifold structure of M and its diameter $d(M)$ of a δ-pinched riemannian manifold.

One of our main results obtained in the present paper is:

A connected and complete riemannian manifold with $\delta = 1/4$ is homeomorphic to a sphere if the diameter $d(M)$ of M satisfies $d(M) > \pi$.

For a simply connected riemannian manifold with $\delta = 1/4$, Klingenberg claimed in [9] that the distance $d(p, C(p))$ between any point $p \in M$ and its cut locus $C(p)$ is no less than π, and M is either homeomorphic to a sphere or M is isometric to one of the compact symmetric spaces of rank one. However the proof stated in [9] seems to us to be incomplete[1].

As the main theorem, it will be proved that *a three dimensional, connected, complete and orientable riemannian manifold with $\delta > 1/4$ is isometric to the lens space $L(1, k)$ of constant curvature 1, if M has a closed geodesic segment Γ with the length $\mathscr{L}(\Gamma) = 2\pi/k$ and the fundamental group $\pi_1(M)$ of M satisfies $\pi_1(M) = Z_k$, where k is an odd prime.*

Definitions and notations are given in § 1. In § 2, we shall give an estimate

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[1] **Added in Proof.** Recently J. Cheeger proved this theorem completely.

of the distance between some point p on a δ-pinched riemannian manifold and its cut locus $C(p)$, which plays an important role in a proof of a sphere theorem stated above, and the sphere theorem will be proved in this section. In § 3, we shall study some estimates of cut loci of δ-pinched riemannian manifolds which are not simply connected. In § 4, we shall investigate some topological structure of a δ-pinched riemannian manifold with $\delta > 1/4$ whose fundamental group satisfies $\pi_1(M) = Z_2$. In the last section, we shall prove our main theorem stated above.

1. Definitions and notations

Throughout this paper let M be a connected, complete and differentiable riemannian manifold of dimension $n (n \geq 2)$, whose sectional curvature K_σ satisfies $0 < \delta \leq K_\sigma \leq 1$ for every plane section σ. Geodesics in M are parametrized by arc-length, and the tangent space at a point $x \in M$ is denoted by M_x. Let u and v be tangent vectors at x, and denote by $\langle u, v\rangle$ the inner product of u and v with respect to the riemann metric tensor of M and by d the distance function of M. For a geodesic segment $\Gamma = \{\gamma(t)\}$ $(0 \leq t \leq l)$, the length of Γ is denoted by $\mathscr{L}(\Gamma)$ which is equal to l. A geodesic triangle (Γ, Λ, Φ) in M is a triple of shortest geodesic segments each of which is not a constant geodesic. For a geodesic triangle (Γ, Λ, Φ) let $(\Gamma^*, \Lambda^*, \Phi^*)$ be the geodesic triangle in $S^2_{1/\sqrt{\delta}}$ satisfying $\mathscr{L}(\Gamma^*) = \mathscr{L}(\Gamma)$, $\mathscr{L}(\Lambda^*) = \mathscr{L}(\Lambda)$ and $\mathscr{L}(\Phi^*) = \mathscr{L}(\Phi)$, where S^k_r denotes the k-sphere with radius r in a euclidean space R^{k+1}. We shall call $(\Gamma^*, \Lambda^*, \Phi^*)$ the corresponding triangle of (Γ, Λ, Φ) in $S^2_{1/\sqrt{\delta}}$. The universal covering manifold of M is denoted by $\tilde{M}$ and the projection map by π. The diameter $d(M)$ of M is defined by $d(M) = \sup\{d(x, y) \mid x, y \in M\}$.

Let G be the cyclic group of order k whose generator g is given by $g = \begin{bmatrix} R(1/k) & \\ & R(1/k) \end{bmatrix}$, where k is an odd prime and $R(\theta)$ means the rotation of R^2 which is defined by $R(\theta) = \begin{bmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$. The lens space $L(1, k)$ of constant curvature 1 is defined by $L(1, k) = S^3_1/G$ where k is an odd prime.

2. An estimate of cut locus of certain δ-pinched manifold

In this section, we shall give an estimate of the distance between some point $x \in M$ and its cut locus $C(x)$ where the diameter $d(M)$ of M satisfies $d(M) > \pi/(2\sqrt{\delta})$. Our technique does not hold for all points of M but for some pair of points $x, y \in M$ satisfying $d(x, y) > \pi/(2\sqrt{\delta})$ for any $0 < \delta \leq 1$.

First of all, we shall prove the following proposition.

Proposition 2.1. *If the diameter $d(M)$ of M satisfies $d(M) > \pi/(2\sqrt{\delta})$ for any $0 < \delta \leq 1$, then M is simply connected.*

Proof. Suppose that M is not simply connected. Let p and q be the points in M such that $d(p, q) = \mathrm{d}(M)$. There are at least two points $\tilde{p}_1$ and $\tilde{p}_2$ in $\tilde{M}$ satisfying $\pi(\tilde{p}_1) = \pi(\tilde{p}_2) = p$. By completeness of $\tilde{M}$, there exists a shortest geodesic $\tilde{\Theta} = \{\tilde{\theta}(t)\}$ $(0 \leq t \leq l)$ satisfying $\tilde{\theta}(0) = \tilde{p}_1$, $\tilde{\theta}(l) = \tilde{p}_2$ and $\mathscr{L}(\tilde{\Theta}) = l = d(\tilde{p}_1, \tilde{p}_2)$. Putting $\Theta = \pi \circ \tilde{\Theta}$, we have a geodesic $\Gamma = \{\gamma(t)\}$ $(0 \leq t \leq d(M))$ such that $\gamma(0) = p$, $\gamma(d(M)) = q$ which satisfies $\langle \gamma'(0), \theta'(0) \rangle \geq 0$, where $\gamma'(t)$ denotes the tangent vector of Γ at $\gamma(t)$. Then, there is a geodesic $\tilde{\Gamma}$ in $\tilde{M}$ which satisfies $\Gamma = \pi \circ \tilde{\Gamma}$ and $\tilde{\gamma}(0) = \tilde{p}_1$, $\tilde{\gamma}(d(M)) = \tilde{q} \in \tilde{M}$. Consider a geodesic triangle $(\tilde{\Gamma}, \tilde{\Theta}, \tilde{\Lambda})$ in $\tilde{M}$ where $\tilde{\Lambda}$ is a shortest geodesic joining $\tilde{q}$ to $\tilde{p}_2$. Assume that the perimeter of $(\tilde{\Gamma}, \tilde{\Theta}, \tilde{\Lambda})$ is less than $2\pi/\sqrt{\delta}$, and let $(\Gamma^*, \Theta^*, \Lambda^*)$ be the corresponding geodesic triangle of $(\tilde{\Gamma}, \tilde{\Theta}, \tilde{\Lambda})$ in $S^2_{1/\sqrt{\delta}}$. Then by virtue of the basic theorem on the triangles of Toponogov, every angle of $(\tilde{\Gamma}, \tilde{\Theta}, \tilde{\Lambda})$ is not less than the corresponding angle of $(\Gamma^*, \Theta^*, \Lambda^*)$. Hence we have $\sphericalangle\, (\gamma^{*\prime}(0), \theta^{*\prime}(0)) \leq \sphericalangle\, (\tilde{\gamma}(0), \tilde{\theta}'(0)) \leq \pi/2$. On the other hand, the inequality $\mathscr{L}(\tilde{\Lambda}) \geq \mathscr{L}(\tilde{\Gamma}) > \pi/(2\sqrt{\delta})$ implies that $\sphericalangle\, (\gamma^{*\prime}(0), \theta^{*\prime}(0)) > \pi/2$, giving a contradiction. Therefore the perimeter of $(\tilde{\Gamma}, \tilde{\Theta}, \tilde{\Lambda})$ must be $2\pi/\sqrt{\delta}$. Then Theorem 4 of [13] implies that M is isometric to the n-sphere $S^n_{1/\sqrt{\delta}}$ of radius $1/\sqrt{\delta}$. Making use of the inequality $\sphericalangle\, (\tilde{\gamma}'(0), \tilde{\theta}'(0)) \leq \pi/2$, was see that $\mathscr{L}(\tilde{\Gamma}) = \mathscr{L}(\tilde{\Lambda}) + \mathscr{L}(\tilde{\Theta}) = \pi/\sqrt{\delta}$ or $\mathscr{L}(\tilde{\Theta}) = \mathscr{L}(\tilde{\Gamma}) + \mathscr{L}(\tilde{\Lambda}) = \pi/\sqrt{\delta}$. If $\mathscr{L}(\tilde{\Gamma}) = \pi/\sqrt{\delta}$, then $\mathscr{L}(\Gamma) = d(M) = \pi/\sqrt{\delta}$ implies that M is isometric to $S^n_{1/\sqrt{\delta}}$. If $\mathscr{L}(\tilde{\Theta}) = \mathscr{L}(\tilde{\Gamma}) + \mathscr{L}(\tilde{\Lambda}) = \pi/\sqrt{\delta}$ holds, we have $\mathscr{L}(\Gamma) \leq \pi/(2\sqrt{\delta})$ from $\mathscr{L}(\tilde{\Gamma}) \leq \mathscr{L}(\tilde{\Lambda})$, which is a contradiction.

Theorem 2.2. *For any pair of points x, y in M satisfying $d(x, y) > \pi/(2\sqrt{\delta})$, we have $d(x, C(x)) \geq \pi$ and $d(y, C(y)) \geq \pi$ where $C(x)$ denotes the cut locus of x.*

Proof. It δ satisfies $\delta > 1/4$, Proposition 2.1 and a theorem of Klingenberg [8] imply the statement. Suppose that $d(y, C(y)) = \rho < \pi$ holds for some pair of points x, y satisfying $d(x, y) > \pi/(2\sqrt{\delta})$. We shall derive a contradiction, and need only to consider δ satisfying $\delta \leq 1/4$. By the hypothesis $\rho < \pi$ and an elementary property of cut locus, there is a closed geodesic segment $\Sigma = \{\sigma(t)\}$ $(0 \leq t \leq 2\rho)$ such that $\sigma(0) = \sigma(2\rho) = y$. For any $t \in [0, 2\rho]$, we get $d(x, \sigma(t)) \geq d(x, y) - d(y, \sigma(t)) > \pi/(2\sqrt{\delta}) - \pi \geq 0$ which shows that $x \notin \Sigma$. Then there exists a point z on Σ satisfying $d(x, z) = d(x, \Sigma)$. Suppose that $z \neq y$. Then by virtue of the second variation formula [1, Proposition 3], we have $d(x, \Sigma) \leq \pi/(2\sqrt{\delta})$. The points y and z divide Σ into two subarcs. Let $\hat{\Sigma}$ be the shorter subarc, Φ and Λ be the shortest geodesics from x to y and x to z respectively, and $(\Phi^*, \hat{\Sigma}^*, \Lambda^*)$ be the corresponding geodesic triangle of $(\Phi, \hat{\Sigma}, \Lambda)$ in $S^2_{1/\sqrt{\delta}}$. Then the inequalities $\mathscr{L}(\Phi^*) > \pi/(2\sqrt{\delta})$, $\mathscr{L}(\hat{\Sigma}^*) \leq \rho < \pi/(2\sqrt{\delta})$ and $\mathscr{L}(\Lambda^*) \leq \pi/(2\sqrt{\delta})$ imply that the angle between Σ^* and Λ^* is greater than $\pi/2$, which contradicts the basic theorem on triangles.

Therefore we must have $y = z$, and we have immediately $d(x, \sigma(t)) > d(x, y)$ for all $t \in (0, 2\rho)$. Putting $y_1 = \sigma(\rho)$ and $d(y_1, C(y_1)) = \rho_1$, we get $\rho_1 \leq \rho$ from $y \in C(y_1)$ and $d(x, y_1) > d(x, y) > \pi/(2\sqrt{\delta})$. There is a closed geodesic segment

$\Sigma_1 = \{\sigma_1(t)\}$ $(0 \leq t \leq 2\rho_1)$ such that $\sigma_1(0) = \sigma_1(2\rho_1) = y_1$ and $x \notin \Sigma_1$ and therefore we have the same argument for Σ_1 as for Σ. If Σ_1 is a closed geodesic, the second variation formula stated above implies that the nearest point $z_1 \in \Sigma_1$ to x is different from y_1, and the same discussion for the geodesic triangle with vertices (x, y_1, z_1) leads a contradiction. Hence we only consider Σ_1 being a closed geodesic segment and satisfying $d(x, \sigma_1(t)) > d(x, y_1)$ for all $t \in (0, 2\rho_1)$.

Putting again $y_2 = \sigma_1(\rho_1)$ and $\rho_2 = d(y_2, C(y_2))$, there is a closed geodesic segment $\Sigma_2 = \{\sigma_2(t)\}$ $(0 \leq t \leq 2\rho_2)$, where we have $\rho_2 \leq \rho_1 \leq \rho < \pi$ and $d(x, y_2) > d(x, y_1) > d(x, y) > \pi/(2\sqrt{\delta})$. Repeating this argument, we have the sequences of points, closed geodesic segments and real numbers as follows:

$$\begin{aligned} &y, y_1, y_2, \cdots, \\ &\Sigma, \Sigma_1, \Sigma_2, \cdots, \\ &\rho \geq \rho_1 \geq \rho_2 \geq, \cdots, \\ &d(x, y) < d(x, y_1) < d(x, y_2) <, \cdots . \end{aligned}$$

Since M is compact, the last sequence satisfies $d(x, y_k) \leq d(M)$ for all k, from which $d(x, y_k)$ has a limit and we can choose a subsequence of $\{y_k\}$ converging to some point y^* in M by compactness. Because the function $p \to d(p, C(p))$ is lower semi-continuous, we have $\lim \rho_k \geq \rho^*$ where $\rho^* = d(y^*, C(y^*))$.

On the other hand, there is a shortest geodesic Φ_{i-1} from x to y_{i-1}, and for any fixed Φ_i we have the subarc $\hat{\Sigma}_{i-1}$ of Σ_{i-1} which starts from y_{i-1} and ends at y_i with the property that the angle between Φ_i and $\hat{\Sigma}_{i-1}$ at y_i is no greater than $\pi/2$. Let $(\Phi^*_{i-1}, \hat{\Sigma}^*_{i-1}, \Phi^*_i)$ be the geodesic triangle corresponding to $(\Phi_{i-1}, \hat{\Sigma}_{i-1}, \Phi_i)$ in $S^2_{1/\sqrt{\delta}}$, where we denote $\Phi_0 = \Phi$ and $\Sigma_0 = \Sigma$, and let α_i be the angle between Φ^*_i and $\hat{\Sigma}^*_{i-1}$. Then we get $\alpha_i \leq \pi/2$ for all i. By the spherical trigonometry, it follows that

$$\begin{aligned} &\cos(d(x, y_{i-1})\sqrt{\delta}) - \cos(d(x, y_i)\sqrt{\delta}) \cdot \cos(\rho_{i-1}\sqrt{\delta}) \\ &\quad = \sin(\rho_{i-1}\sqrt{\delta}) \cdot \sin(d(x, y_i)\sqrt{\delta}) \cdot \cos\alpha_i \geq 0 , \end{aligned}$$

which implies $\cos(d(x, y_{i-1})\sqrt{\delta}) \geq \cos(d(x, y_i)\sqrt{\delta}) \cdot \cos(\rho_{i-1}\sqrt{\delta})$, for all i. Therefore it follows clearly that

$$\begin{aligned} &\cos(d(x, y)\sqrt{\delta}) \geq \cos(d(x, y_1)\sqrt{\delta}) \cdot \cos(\rho\sqrt{\delta}) \geq \cos(d(x, y_k)\sqrt{\delta}) \\ &\cdot \prod_{i=1}^{k} \cos(\rho_{i-1}\sqrt{\delta}) \geq \cos(d(x, y_k)\sqrt{\delta}) \cdot (\cos(\rho^*\sqrt{\delta}))^k , \qquad k = 1, 2, \cdots . \end{aligned}$$

Hence we must have $\cos(d(x, y)\sqrt{\delta}) \geq 0$, so that $d(x, y) \leq \pi/(2\sqrt{\delta})$, a contradiction. q.e.d.

In order to estimate the distance betweem a point $p \in M$ and its cut locus $C(p)$, the simply connectedness of M is the essential hypothesis for the arguments developed in [7], [8] and [9]. We note that the technique of a proof of

Sphere Theorem investigated by Klingenberg need not the estimate $d(x, C(x)) \geq \pi$ for all points of M.

Theorem 2.3. *Let M be a connected and complete riemannian manifold. If the sectional curvature K_σ of M satisfies $1/4 \leq K_\sigma \leq 1$ for every plane section σ and the diameter $d(M)$ of M satisfies $d(M) > \pi$, then M is homeomorphic to S^n.*

By virtue of Theorem 2.2, it suffices to show the following proposition for a proof of Theorem 2.3.

Proposition 2.4. *Suppose that $\delta = 1/4$ and $d(M) > \pi$ hold, and set $d(p, q) = d(M)$. Then for any point $r \in M$, we have $d(p, r) < \pi$ or $d(q, r) < \pi$.*

In the following we prepare Lemmas 2.5–2.8 for a proof of Proposition 2.4. The method is analogous to that of Berger [3].

Lemma 2.5 (*Lemma* 4 *of Berger* [3]). *For any point $r \in M$, we have $d(p, r) < \pi$ or $d(q, r) < \pi$ or otherwise $d(p, r) = d(q, r) = \pi$.*

Lemma 2.6 (*Lemma* 5 *of Berger* [3]). *Suppose that there is a point $r \in M$ satisfying $d(p, r) = d(q, r) = \pi$, where $d(p, q) = d(M)$. For any shortest geodesic $\Phi = \{\varphi(t)\}$ $(0 \leq t \leq \pi)$, $\varphi(0) = p$, $\varphi(\pi) = r$, let Γ be a geodesic such that $\Gamma = \{\gamma(t)\}$ $(0 \leq t \leq d(M))$, $\gamma(0) = p$, $\gamma(d(M)) = q$ and $\measuredangle\, (\gamma'(0), \varphi'(0) \leq \pi/2$. Then we have $d(r, \gamma(t)) = \pi$ for all $0 \leq t \leq d(M)$ and there is a piece of totally geodesic surface of constant curvature $1/4$ with boundaries Φ, Γ and Ψ, where Ψ is a geodesic such that $\Psi = \{\Psi(t)\}$ $(0 \leq t \leq \pi)$, $\psi(0) = q$, $\psi(\pi) = r$, and we also have $\measuredangle\, (\varphi'(0), \gamma'(0)) = \pi/2$, $\measuredangle\, (\gamma'(d(M)), \psi'(0)) = \pi/2$ and $\measuredangle\, (\varphi'(\pi), \psi'(\pi)) = d(M)/2$.*

We can prove Lemmas 2.5 and 2.6 in the same way as that stated in [3].

Lemma 2.7. *Let N be defined by $N = \{x \in M \mid d(x, y) > \pi/(2\sqrt{\delta})$ for some $y \in M\}$ where δ is any positive constant $0 < \delta \leq 1$. For any fixed point $x \in N$, let Θ and Θ_1 be shortest geodesics of length π satisfying $x = \theta(0) = \theta_1(0)$, $\theta(\pi) = \theta_1(\pi) = z$ and $\theta'(0) \neq \pm\theta_1'(0)$. Then there exists a lune of totally geodesic surface of constant curvature 1 with boundaries Θ and Θ_1.*

Proof. $\theta'(0) \neq \pm\theta_1'(0)$ implies clearly $\theta'(\pi) \neq \pm\theta_1'(\pi)$ from Theorem 2.2. Since N is open in M there is a point $w \in \Theta \cap N$. It follows that $d(w, \theta_1(t)) < \pi$ for every $t \in [0, \pi]$ and $\exp_w | U$ is a diffeomorphism, where U is an open ball in M_w with radius π and center at the origin.

Let $\Theta^* = \{\theta^*(t)\}$ $(0 \leq t \leq \pi)$ be a great circle on S_1^n. Take a point $w^* \in \Theta^*$ satisfying $w^* = \theta^*(t_0)$, where t_0 is defined by $w = \theta(t_0)$. Let ι be an isometric isomorphism of M_w onto $(S_1^n)_{w^*}$ such that $\iota \circ \theta'(t_0) = \theta^{*\prime}(t_0)$ and put $x^* = \theta^*(0)$, $z^* = \theta^*(\pi)$. Then the curve $\Theta_1^* = (\exp_{w^*} \circ \iota \circ (\exp_w | U)^{-1}) \circ \Theta_1$ is a regular curve which connects x^* to z^* and whose length is equal to π by Rauch's metric comparison theorem [11]. Since Θ_1^* becomes a great circle, we obtain a *lune* of totally geodesic surface of constant curvature 1 with boundaries Θ and Θ_1.

Lemma 2.8 (*Lemma* 6 *of Berger* [3]). *Let M be a riemannian manifold whose sectional curvature K_σ satisfies $\delta \leq K_\sigma \leq 1$ for every plane section σ,* *and let X, Y and Z be tanget vectors at $x \in M$ such that $Y \neq Z$, $K(X, Y) = K(X, Z) = \delta$ and $\langle Y, X\rangle = \langle Z, X\rangle > 0$. Then we have $K(Y, Z) < 1$.*

Proof of Theorem 2.3. Let Γ, Φ, Θ be the shortest geodesic segments joining p to q, p to r and q to r respectively which are defined in such a way that there is a piece of totally geodesic surface $\mathscr{F}$ of constant curvature $1/4$ with boundaries Γ, Φ and Θ. Developing the same discussion as Berger [3], there is a shortest geodesic $\Gamma_1 = \{\gamma_1(t)\}$ $(0 \leq t \leq d(M))$, $\gamma_1(0) = p$, $\gamma_1(d(M)) = q$ which satisfies $\Gamma_1 \neq \Gamma$ and $\langle \gamma_1'(0), \varphi'(0)\rangle = 0$. Therefore we have another totally geodesic surface $\mathscr{F}_1$ of constant curvature $1/4$ with boundaries Γ_1, Φ and Θ_1, where Θ_1 is a shortest geodesic from q to r. Suppose that $\sphericalangle\, (\theta'(0), \theta_1'(0)) = \pi$. We have $\sphericalangle\, (\theta'(\pi), \theta_1'(\pi)) = \pi$ and moreover $\mathscr{F}$ and $\mathscr{F}_1$ have the same tangent plane at r. Hence we get $\sphericalangle(\varphi'(\pi), \theta'(\pi)) = \sphericalangle\, (\varphi'(\pi), \theta_1'(\pi)) = \pi/2$, which imply $\mathrm{d}(M) = \pi$. Therefore we must have $\sphericalangle\, (\theta'(0), \theta_1'(0)) < \pi$. Suppose that $\sphericalangle\, (\theta'(0), \theta_1'(0)) = 0$. Then $\mathscr{F}$ and $\mathscr{F}_1$ have a common tangent plane at q from which we get $\Gamma = \Gamma_1$. Hence we have $\sphericalangle\, (\theta'(0), \theta_1'(0)) \in (0, \pi)$, and Lemma 2.8 implies a contradiction.

3. Estimates of cut locus of δ-pinched manifold which is not simply connected

In this section we shall investigate some estimates of cut locus of δ-pinched riemannian manifold which is not simply connected.

Proposition 3.1. *If M is not simply connected and $0 < \delta \leq 1$, then $d(p, C(p)) \leq d(M) \leq \pi/(2\sqrt{\delta})$ for every point $p \in M$. Suppose that there is a point $p \in M$ at which $d(p, C(p)) = \pi/(2\sqrt{\delta})$ holds. Then M is isometric to the real projective space $PR^n(\delta)$ of constant curvature δ.*

Proof. Let $\tilde{M}$ be the universal covering manifold of M and π be the projection map. There and at least two distinct points $\tilde{p}_1, \tilde{p}_2$ of $\tilde{M}$ such that $\pi(\tilde{p}_1) = \pi(\tilde{p}_2) = p$. Let $\tilde{\Gamma}$ be a shortest geodesic joining $\tilde{p}_1$ to $\tilde{p}_2$, and Γ be a closed geodesic segment at p defined by $\pi \circ \tilde{\Gamma} = \Gamma$. Then $\mathscr{L}(\Gamma)$ is not less than $2d(p, C(p)) = \pi/\sqrt{\delta}$, from which we have $d(\tilde{p}_1, \tilde{p}_2) \geq \pi/\sqrt{\delta}$. Thus $\tilde{M}$ is isometric to $S^n_{1/\sqrt{\delta}}$ by the maximal diameter theorem of Toponogov [13]. Suppose that there is a point $\tilde{p}_3 \in \tilde{M}$ satisfying $\tilde{p}_1 \neq \tilde{p}_3 \neq \tilde{p}_2$ and $\pi(\tilde{p}_3) = p$. Then the perimeter of a geodesic triangle in M with vertices $\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ is not less than $3\pi/\sqrt{\delta}$, which is a contradiction. Therefore $\tilde{M}$ must be a double covering of M, and hence we get $M = PR^n(\delta)$.

Proposition 3.2. *Let M be a δ-pinched riemannian manifold which is not simply connected, and suppose that there is a point $p \in M$ at which $d(p, C(p)) \geq \pi/(3\sqrt{\delta})$ holds. Then the fundamental group of M is $\pi_1(M) = Z_2$ or otherwise, M is odd dimensional riemannian manifold of constant curvature δ.*

Proof. If there are three different points $\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ in $\tilde{M}$ such that $\pi(\tilde{p}_1) = \pi(\tilde{p}_2) = \pi(\tilde{p}_3) = p \in M$, then we have $d(\tilde{p}_i, \tilde{p}_j) \geq 2d(p, C(p)) \geq 2\pi/(3\sqrt{\delta})$ for every $i, j = 1, 2, 3$, $i \neq j$, and the perimeter of a geodesic triangle with vertices

$\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ is not less than $2\pi/\sqrt{\delta}$, from which M is of constant curvature δ. As is well known, an even dimensional complete riemannian manifold of constant positive curvature is isometric to either a sphere or a real projective space. Hence $\dim M$ must be odd.

Corollary to Proposition 3.2. *Let M be a δ-pinched riemannian manifold which is not simply connected, and suppose that $\pi_1(M) \neq Z_2$. Then we have $d(x, C(x)) \leq \pi/(3\sqrt{\delta})$ for any $x \in M$. Furthermore if there is a point $x \in M$ at which $d(x, C(x)) = \pi/(3\sqrt{\delta})$, then M is an odd dimensional riemannian manifold of constant curvature δ.*

We shall give some estimates of cut loci under the assumption $\delta > 1/4$ and certain assumptions for the fundamental groups of pinched manifolds.

Theorem 3.3. *Suppose that δ satisfies $\delta > 1/4$ and M is not simply connected. Then there does not exist any δ-pinched riemannian manifold M whose diameter satisfies $\pi/(2\sqrt{\delta}) < d(M) < \pi$. In particular if the fundamental group $\pi_1(M)$ of such M satisfies $\pi_1(M) = Z_k$, where k is an integer not less than 2, then we have $d(x, C(x)) \geq \pi/k$ for every point $x \in M$. Moreover if there is a point $x \in M$ where $d(x, C(x)) = \pi/k$ is satisfied, then M is of constant curvature 1.*

Proof. The first statement is evident from Proposition 2.1 and the Theorem of Klingenberg [8]. Suppose that the fundamental group $\pi_1(M)$ of M satisfies $\pi_1(M) = Z_k$, where k is an integer such that $k \geq 2$. Since the function $p \to d(p, C(p))$ is lower semi-continuous, there is a point $p_0 \in M$ at which the function takes infimum ρ. We have a closed geodesic $\Sigma = \{\sigma(t)\}$ $(0 \leq t \leq 2\rho)$ such that $\sigma(0) = \sigma(2\rho) = p_0$ and $\rho = d(p_0, C(p_0))$. Then there exists a closed geodesic $\tilde{\Sigma}$ in $\tilde{M}$ satisfying $\pi \circ \tilde{\Sigma} = \Sigma$. By virtue of $\pi_1(M) = Z_k$, we have $\mathscr{L}(\tilde{\Sigma}) = 2\rho k$. On the other hand, every closed geodesic segment in $\tilde{M}$ has length no less than 2π. Hence we have $\rho \geq \pi/k$.

If there is a point $x \in M$ at which $d(x, C(x)) = \pi/k$ holds. Then we have $\rho = \pi/k$ and $\mathscr{L}(\tilde{\Sigma}) = 2\pi$. We shall prove that $C(\tilde{p}_0)$ consists of only one point $\{\tilde{\sigma}(\pi)\}$. In fact, if there is a point $\tilde{q}$ in $C(\tilde{p}_0)$ such that $\tilde{q} \neq \tilde{\sigma}(\pi)$, then let $\tilde{\Psi}$ be a shortest geodesic from $\tilde{\sigma}(\pi)$ to $\tilde{q}$. Without loss of generality we can assume that $\langle \tilde{\sigma}'(\pi), \tilde{\psi}'(0) \rangle \geq 0$. For a geodesic triangle $(\tilde{\Sigma} | [\pi, 2\pi], \tilde{\Psi}, \tilde{\Phi})$ with vertices $\tilde{p}_0, \tilde{\sigma}(\pi)$ and $\tilde{q}$, we have a contradiction to the basic theorem on triangles, because $\pi > \pi(2\sqrt{\delta})$ holds.

Remark. If the diameter $d(M)$ of M with $\delta > 1/4$ satisfies $d(M) = \pi$, then M is isometric to S_1^n. Furthermore, if there is a closed geodesic segment of length 2π in such a simply connected M, then M is isometric to S_1^n.

4. Topological structures of M satisfying $\delta > 1/4$ and $\pi_1(M) = Z_2$

Throughout this section we only consider M satisfying $\delta > 1/4$ and $\pi_1(M) = Z_2$ First of all we shall prove the following lemma.

Lemma 4.1. *Take a pair of points $p, q \in M$ such that $d(p, q) = d(M)$. Then* *there is a closed geodesic* $\Gamma=\{\gamma(t)\}$ $(0\leq t\leq 2d(M))$ *such that* $\gamma(0)=\gamma(2d(M))$ $=p$ *and* $\gamma(d(M))=q$.

Proof. From the assumptions $\delta>1/4$ and $\pi_1(M)=Z_2$, we have $d(M)\leq \pi/(2\sqrt{\delta})<\pi$. Let $\Gamma=\{\gamma(t)\}$ $(0\leq t\leq d(M))$ be a shortest geodesic from p to q. Since $d(p,q)=d(M)$, there is a shortest geodesic Γ_1, $\Gamma_1=\{\gamma_1(t)\}$ $(0\leq t \leq d(M))$ from p to q satisfying $\langle\gamma'(d(M)),-\gamma_1'(d(M))\rangle\geq 0$. Suppose that $\measuredangle\,(\gamma'(d(M),\gamma_1'(d(M)))\neq\pi$. Then there is a shortest geodesic Γ_2 from p to q satisfying $\langle\gamma'(d(M))+\gamma_1'(d(M)),-\gamma_2'(d(M))\rangle\geq 0$. Take a fixed point $\tilde{p}_1\in M$ such that $\pi(\tilde{p}_1)=p$, and let $\tilde{\Gamma},\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be geodesics in $\tilde{M}$ which satisfy $\pi\circ\tilde{\Gamma}=\Gamma$, $\pi\circ\tilde{\Gamma}_1=\Gamma_1$ and $\pi\circ\tilde{\Gamma}_2=\Gamma_2$, and start from $\tilde{p}_1$. Since there are just two points in $\tilde{M}$, whose images under π are q, we may consider that $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have same extremals. But we have $d(p_1,C(p_1))\geq\pi$ by the theorem of Klingenberg [8]; this is a contradiction. Therefore we must have $\measuredangle\,(\gamma'(d(M),\gamma_1'(d(M)))=\pi$ and $\measuredangle\,(\gamma'(0),\gamma_1'(0))=\pi$. q.e.d.

Making use of Lemma 4.1, we have the following:

Theorem 4.2. *For any point* $x\in M$, $\pi/2\leq d(x,C(x))\leq\pi/(2\sqrt{\delta})$ *and* $\pi/2\leq d(M)\leq\pi/(2\sqrt{\delta})$, *where the left hand side equalities hold if and only if M is isometric to the real projective space* $PR^n(1)$ *of constant curvature* 1, *and the right hand side equalities hold if and only if M is isometric to* $PR^n(\delta)$ *of constant curvature* δ.

Proof. It suffices to prove that M is isometric to $PR^n(\delta)$ if $d(M)=\pi/(2\sqrt{\delta})$. Putting $d(p,q)=d(M)=\pi/(2\sqrt{\delta})$, there is a closed geodesic $\Gamma=\{\gamma(t)\}$ $(0\leq t\leq\pi/\sqrt{\delta})$ satisfying $\gamma(0)=\gamma(\pi/\sqrt{\delta})=p$ and $\gamma(\pi/(2\sqrt{\delta}))=q$. Let $\tilde{\Gamma}$ be the closed geodesic in $\tilde{M}$ defined by $\pi\circ\tilde{\Gamma}=\Gamma$. Then $\tilde{\Gamma}$ becomes a closed geodesic with length $2\pi/\sqrt{\delta}$, and we can decompose $\tilde{\Gamma}$ into four shortest geodesic segments whose lengths are not equal at the same time. A theorem investigated by Sugimoto in [12] thus shows that $\tilde{M}$ is isometric to $S^n_{1/\sqrt{\delta}}$, and hence M is isometric to $PR^n(\delta)$. q.e.d.

Now we shall investigate the topology of M satisfying $\pi/2<d(M)<\pi/(2\sqrt{\delta})$. According to the homology theory, M has the same homology group as that of PR^n under our assumptions $\pi_1(M)=Z_2$ and $\tilde{M}$ is homeomorphic to S^n.

There is an interesting problem which is not yet solved completely.

Problem. Let j be a homeomorphism of S^n onto itself satisfying:

(1) j is fixed point free,

(2) j is involutive.

Then, is S^n/j homeomorphic to PR^n?

Livesay proved this problem affirmatively in [10] under the assumption $n\leq 3$. When j is a diffeomorphism or a piecewise linearly diffeomorphism, Hirsch and Milnor showed in [6] that S^n/j is not diffeomorphic or piecewise linearly diffeomorphic to PR^n in general.

Turning to our situation that $\delta>1/4$ and $\pi_1(M)=Z_2$, we shall prove that M is homeomorphic to S^n/j, where j is a homeomorphism of S^n onto itself with the properties (1) and (2) stated above. For the construction of j, we prepare Lemmas 4.3–4.6 below. We set $d(p, q) = d(M) = l$. and the closed geodesic $\Gamma = \{\gamma(t)\}$ $(0 \leq t \leq 2l)$, $\gamma(0) = \gamma(2l) = p$, and $\gamma(l) = q$ as stated in Lemma 4.1. Then there exists a closed geodesic $\tilde{\Gamma}$ in $\tilde{M}$ satisfying $\pi \circ \tilde{\Gamma} = \Gamma$, and therefore we have $\mathscr{L}(\tilde{\Gamma}) = 4l$.

Lemma 4.3. *Putting $\tilde{p}_1 = \tilde{\gamma}(0)$, $\tilde{q}_1 = \tilde{\gamma}(l)$, $\tilde{p}_2 = \tilde{\gamma}(2l)$ and $\tilde{q}_2 = \tilde{\gamma}(3l)$, for any point $\tilde{x} \in \tilde{M}$ we have $d(\tilde{x}, \tilde{p}_1) \leq \pi/(2\sqrt{\delta})$ or $d(\tilde{x}, \tilde{p}_2) \leq \pi/(2\sqrt{\delta})$.*

Proof. We may suppose that $\tilde{x} \notin \tilde{\Gamma}$. Take a point $\tilde{z}$ on $\tilde{\Gamma}$ satisfying $d(\tilde{x}, \tilde{\Gamma}) = d(\tilde{x}, \tilde{z})$. It follows $d(\tilde{x}, \tilde{z}) \leq \pi/(2\sqrt{\delta})$ by use of the second variation formula (Proposition 3 of [1]). Without loss of generality we may also suppose that $d(\tilde{p}_1, \tilde{z}) \leq l < \pi/(2\sqrt{\delta})$. Making use of the basic theorem on triangles for a geodesic triangle with vertices $(\tilde{p}_1, \tilde{z}, \tilde{x})$, we thus have $d(\tilde{p}_1, \tilde{x}) \leq \pi/(2\sqrt{\delta})$.

q.e.d.

Now, let U_1 and U_2 be open balls with radius π centered at the origin in $\tilde{M}_{\tilde{p}_1}$ and $\tilde{M}_{\tilde{p}_2}$ respectively. Then $\exp_{\tilde{p}_i} | U_i$ is a diffeomorphism. Let D be the standard n-cell with boundary $D = S^{n-1} \subset R^n$, and let V_1 and V_2 be given as follows:

$$V_1 = \{\tilde{x} \in \tilde{M} \,|\, d(\tilde{x}, \tilde{p}_1) \leq d(\tilde{x}, \tilde{p}_2)\} , \qquad V_2 = \{\tilde{x} \in \tilde{M} \,|\, d(\tilde{x}, \tilde{p}_1) \geq d(\tilde{x}, \tilde{p}_2)\} .$$

We have a construction of a homeomorphism h of S^n onto $\tilde{M}$ investigated by Klingenberg in [7] as follows.

Lemma 4.4. *There are homeomorphisms h_1 and h_2 such that $h_i: D \to V_i$ satisfying $h_i(D) = V_i$, $h_1(D) \cup h_2(D) = M$ and $h_1(D) \cap h_2(D) = h_1(S^{n-1}) = h_2(S^{n-1})$. Making use of h_1 and h_2, we have a homeomorphism h: $S^n \to \tilde{M}$.*

On the other hand, by virtue of the hypothesis $\pi_1(M) = Z_2$ we have a map f of $\tilde{M}$ onto itself defined by $f(\tilde{x}_1) = \tilde{x}_2$ for any $\tilde{x}_1 \in \tilde{M}$, where $\pi(\tilde{x}_1) = \pi(\tilde{x}_2)$, $\tilde{x}_1 \neq \tilde{x}_2$. Then clearly we have the following:

Lemma 4.5. *f satisfies the following:*

(a) *f is an isometry.*

(b) *f is involutive.*

(c) *f has no fixed point.*

(d) *$f \circ \tilde{\Gamma} = \tilde{\Gamma}$, where $\tilde{\Gamma}$ is stated in Lemma* 4.3.

Combining Lemmas 4.4 and 4.5, we get

Lemma 4.6. *We have $f(V_1) = V_2$ and $f(V_2) = V_1$. In particular $f(\tilde{p}_1) = \tilde{p}_2$.*

Proof. $f \circ \tilde{\Gamma} = \tilde{\Gamma}$ implies $f(\tilde{p}_1) = \tilde{p}_2$. Take a point $\tilde{x} \in V_1 \cap V_2$. Then there exist uniquely determined shortest geodesics $\tilde{\Lambda}$ and $\tilde{\Phi}$ joining $\tilde{p}_1$ to $\tilde{x}$ and $\tilde{p}_2$ to $\tilde{x}$ respectively. Thus $\tilde{\Lambda}$ and $\tilde{\Phi}$ have the same length which is not greater than $\pi/(2\sqrt{\delta})$, an dthe intersection of $f \circ \tilde{\Lambda}$ and $f \circ \tilde{\Phi}$ must coincide with $f(\tilde{x})$. Hence we get $f(\tilde{x}) \in V_1 \cap V_2$, from which the statements follow. q.e.d.

Combining Lemmas 4.3–4.6, we find the following:

Theorem 4.7. *Let j be defined by $j = h^{-1} \circ f \circ h$. Then M is homeomorphic to S^n/j, and j satisfies* (1) *and* (2) *in the problem stated above.*

Remark. According to [10], Livesay proved that S^n/j is homeomorphic to PR^n if $n \leq 3$. But in our case, we shall be able to prove that M is homeomorphic to PR^n if $n \leq 4$. Since $V_1 \cap V_2$ is homeomorphic to PR^3 (in case $n = 4$), (c) in Lemma 4.5 implies the statement.

Putting $p_i^* = h^{-1}(\tilde{p}_i)$, p_i^* is the antipodal point of p_2^* on S_1^n. Hence the image of every great circle from p_1^* to p_2^* under j is also a great circle from p_2^* to p_1^*.

5. Proof of the main theorem

Throughout this section, let k be an odd prime. Let M be a δ-pinched ($\delta > 1/4$) riemannian manifold whose fundamental group $\pi_1(M)$ satisfies $\pi_1(M) = Z_k$. Then we shall prove the following:

Theorem 5.1. *Let M be a connected, complete and orientable riemannian manifold of dimension 3 satisfying $\delta > 1/4$ and $\pi_1(M) = Z_k$, and suppose that there is a closed geodesic segment Γ of length $2\pi/k$. Then M is isometric to the lens space $L(1, k)$ of constant curvature 1.*

Our method of the proof is as follows:

Put $M^* = L(1, k)$ and take two arbitrarily fixed points $p^* \in M^*$ and $p \in M$ respectively. It is clear that M is of constant curvature 1. It is easily seen that for any tangent vector $X^* \in M^*_{p^*}$ satisfying $X^* \in C_{p^*}$, we have $X^* \notin Q^*_{p^*}$, where $Q^*_{p^*}$ is the first conjugate locus in $M^*_{p^*}$. Then there is at least one tangent vector $Y^* \in C^*_{p^*}$ which satisfies $\exp_{p^*} X^* = \exp_{p^*} Y^* \in C(p^*)$. We shall prove that there is an isometric isomorphism ι of M_p onto $M^*_{p^*}$ such that $\iota(C_p)$ concides with $C^*_{p^*} \subset M^*_{p^*}$ as a set in $M^*_{p^*}$, and moreover the identifying structures of C_p under $\exp_p$ and $C^*_{p^*}$ under $\exp_{p^*}$ are quite equivalent under ι. That is to say, let $X, Y \in C_p$ and $\exp_p X = \exp_p Y \in C(p)$. Then we have $\exp_{p^*} \iota \circ X = \exp_{p^*} \iota \circ Y \in C^*(p^*)$. Hence $\exp_{p^*} \circ \iota \circ \exp_p^{-1}$ becomes a global isometry of M onto M^*.

As the first step, we study the tangent cut lous C_p of M. Theorem 3.3 and the hypothesis of M imply that M is of constant curvature 1. Then the universal covering manifold $\tilde{M}$ is S_1^3.

Lemma 5.2. *Let M satisfy the assumptions of Theorem 5.1. Then $d(q, C(q)) = \pi/k$ for any point $q \in M$.*

Putting $l = d(q, C(q))$, there is a closed geodesic segment Σ_q of length $2l$ such that $\sigma_q(0) = \sigma_q(2l) = q$. Then we have a great circle $\tilde{\Sigma}$ in $S_1^3 = M$ satisfying $\pi \circ \tilde{\Sigma} = \Sigma_q$, on which we get $\pi(\tilde{\sigma}_q(0)) = \pi(\tilde{\sigma}_q(2l)) = \cdots = \pi(\tilde{\sigma}_q(2kl)) = q$. Hence we have $2kl = 2\pi$. q.e.d.

We denote by Σ_q the closed geodesic at q with length $2\pi/k$.

Lemma 5.3. Max $\{d(q, x) \mid x \in M\} = \pi/2$ *for any point $q \in M$. In particular, $d(M) = \pi/2$.*

Proof. Putting $l = d(q, r) = \text{Max}\,\{d(q, x) \mid x \in M\}$, there is a closed geodesic $\Sigma_r = \{\sigma_r(t)\}$ $(0 \leq t \leq 2\pi/k)$ such that $\sigma_r(0) = \sigma_r(2\pi/k) = r$. By the assumption of $d(q, r)$, there are at least two shortest geodesic segments joining q to r, say Γ_1 and Γ_2. Suppose that $\measuredangle\,(\gamma_1'(l), \gamma_2'(l)) = \pi$. Since $l \leq d(M) \leq \pi/(2\sqrt{\delta})$

$= \pi/2$ and k is an odd prime, there exist at least $k+1$ points on S_1^3 whose images under π are all q. Then we must have $\measuredangle(\gamma_1'(l), \gamma_2'(l)) \neq \pi$, from which there is another shortest geodesic Γ_3 from q to r such that $\langle \gamma_1'(l) + \gamma_2'(l), -\gamma_3'(l) \rangle \geq 0$. Let $\tilde{q} \in \tilde{M}$ be a fixed point such that $\pi(\tilde{q}) = q$, and $\tilde{\Gamma}_i$ be defined by $\pi \circ \tilde{\Gamma}_i = \Gamma_i$ and $\tilde{\gamma}_i(0) = \tilde{q}$ $(i = 1, 2, 3)$. It is clear that the geodesic $\tilde{\Sigma}$ given by $\pi \circ \tilde{\Sigma} = \Sigma_r$ is a great circle on which lie the points $\tilde{\gamma}_1(l), \tilde{\gamma}_2(l)$ and $\tilde{\gamma}_3(l)$. Three geodesic triangles with vertices $(\tilde{q}, \tilde{\gamma}_1(l), \tilde{\gamma}_2(l))$, $(\tilde{q}, \tilde{\gamma}_2(l), \tilde{\gamma}_3(l))$ and $(\tilde{q}, \tilde{\gamma}_3(l), \tilde{\gamma}_1(l))$ respectively become isosceles triangles whose base angles are all equal to $\pi/2$. Therefore we must have $l = \pi/2$ by the cosine rule of spherical trigonometry.

Lemma 5.4. *Let $q, p \in M$ be a fixed pair of points such that $d(p, q) = \pi/2$. Then there are shortest geodesics $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ from p to q satisfying the following:*

(1) $\measuredangle(\gamma_i'(0), \gamma_{i+1}'(0)) = \measuredangle(\gamma_i'(\pi/2), \gamma_{i+1}'(\pi/2)) = 2\pi/k$ *for all* $i = 1, 2, \cdots, k$, (mod k).

(2) *There is a piece of totally geodesic surface $\mathcal{F}_i^+$ of constant curvature* 1 *whose boundaries are Γ_i, Γ_{i+1} and Σ_p.*

(3) *It can be considered that $\mathcal{F}_i^+$ is generated by the family of shortest geodesics $\{\Lambda_t\}$ $(0 \leq t \leq 2\pi)$ where each Λ_t starts from $\sigma_p(t)$ and ends at q with length $\mathcal{L}(\Lambda_t) = \pi/2$. Moreover, we can consider that $\Lambda_0 = \Gamma_1$ and $\Lambda_{2\pi(k-1)/k} = \Gamma_k$, and the vector field $t \to \lambda_t'(0)$ is parallel along Σ_p.*

(4) *Putting $\pi \circ \tilde{\Gamma}_i = \Gamma_i$ such that $\tilde{\gamma}_i(\pi/2) = \tilde{q}$ where $\pi(\tilde{q}) = q$, each $\mathcal{F}_i^+$ is covered by the face of geodesic triangle $(\tilde{\Gamma}_i, \tilde{\Gamma}_{i+1}, \tilde{\Sigma}_{\tilde{p}} \,|\, [2\pi(i-1)/k, 2\pi i/k])$ under the covering map π, where $\pi \circ \tilde{\Sigma}_{\tilde{p}} = \Sigma_p \tilde{\sigma}_{\tilde{p}}(0) = \tilde{p}$. In particular, $\mathcal{F}_1^+ \cup \mathcal{F}_2^+ \cup \cdots \cup \mathcal{F}_k^+$ is the image of the two dimensional hemisphere with north pole $\tilde{q}$ and equator $\tilde{\Sigma}_{\tilde{p}}$ under π.*

Proof. Let $S^2(\tilde{q})$ be the totally geodesic hypersurface of S_1^3, which contains $\tilde{q}$ and $\tilde{\Sigma}_{\tilde{p}}$, and $S_+^2(q)$ be the hemisphere with north pole $\tilde{q}$. For a geodesic segment Λ_t in $S_+^2(q)$ joining $\tilde{\sigma}_{\tilde{p}}(t)$ to $\tilde{q}$ and the corresponding geodesic $\Lambda_t = \pi \circ \tilde{\Lambda}_t$ in M joining $\sigma_p(t)$ to q, making use of Rauch's comparison theorem we get the statements (2), (3) and (4). Since we have $\measuredangle(\tilde{\gamma}_i'(\pi/2), \tilde{\gamma}_{i+1}'(\pi/2)) = \measuredangle(\gamma_i'(\pi/2), \gamma_{i+1}'(\pi/2)) = 2\pi/k$ for $i = 2, 3, \cdots, k-1$, we get $\measuredangle(\gamma_i'(0), \gamma_{i+1}'(0)) = 2\pi/k$ by exchanging the situation of p for the one of q. q.e.d.

Let us put $\mathcal{F}^+ = \mathcal{F}_1^+ \cup \mathcal{F}_2^+ \cup \cdots \cup \mathcal{F}_k^+$. Since $d(p, \sigma_q(\pi/k)) = \pi/2$ holds, $\sigma_q(\pi/k)$ is able to take place for q in the Lemma 5.2–5.4. Then we have a piece of totally geodesic hypersurface $\mathcal{F}_i^-$ of constant curvature 1 with boundaries $\Gamma_i \,|\, [-\pi/2, 0]$, $\Gamma_{i+1} \,|\, [-\pi/2, 0]$ and Σ_p which is a prolongation of $\mathcal{F}_i^+$. Putting $\mathcal{F}^- = \mathcal{F}_1^- \cup \mathcal{F}_2^- \cup \cdots \cup \mathcal{F}_k^-$, we get a compact totally geodesic hypersurface $\mathcal{F}^{q,p} = \mathcal{F}^+ \cup \mathcal{F}^-$ which is the image $\pi(S^2(q))$ of $S^2(q) \subset S_1^3$ under the covering map π. It is clearly seen that $\mathcal{F}^{q,p}$ covers $\Sigma_p k$ times, and its tangent space $(\mathcal{F}^{q,p})_p$ at p consists of k-sheeted planes $(\mathcal{F}_1^+ \cup \mathcal{F}_1^-)_p, \cdots, (\mathcal{F}_k^+ \cup \mathcal{F}_k^-)_p$ each of which contains $\sigma_p'(0)$ and the angle between $(\mathcal{F}_i^+ \cup \mathcal{F}_i^-)_p$ and $(\mathcal{F}_{i+1}^+ \cup \mathcal{F}_{i+1}^-)_p$ is equal to $2\pi/k$.

Lemma 5.5. *The cut locus of the totally geodesic hypersurface* $\pi(S^2(\tilde{q})) = \mathscr{F}^{q,p}$ *with respect to p consists of* $\Lambda_{\pi/k}|[-\pi/2, \pi/2]$, $\Lambda_{3\pi/k}|[-\pi/2, \pi/2]$ *and* $\Lambda_{(2k-1)\pi/k}|[-\pi/2, \pi/2]$, *which is contained entirely in the cut locus $C(p)$ of M.*

Proof. By the construction of $\mathscr{F}^{q,p}$, the first statement is evident. Suppose that there is a shortest geodesic of M from p to $\lambda_{\pi/k}(s) \in \mathscr{F}^{q,p}$ which is not contained in $\mathscr{F}^{q,p}$. Then there are at least $k+1$ points in S_1^3 whose images under π are $\lambda_{\pi/k}(s)$. Hence p and $\lambda_{\pi/k}(s)$ can be joined by shortest geodesics of M which lie in $\mathscr{F}^{q,p}$. q.e.d.

By exchanging q (north pole) and Σ_p (equator) for p and Σ_q respectively, we get a compact totally geodesic surface $\mathscr{F}^{p,q}$ instead of $\mathscr{F}^{q,p}$ whose tangent space $(\mathscr{F}^{p,q})_p$ at p is the plane in M_p orthogonal to $\sigma_p'(0)$. Therefore we get the family of compact totally geodesic hypersurfaces $\{\mathscr{F}^{\sigma_q(t),p}\}$ $(0 \leq t \leq 2\pi)$, and M can be considered to be constructed by this family of hypersurfaces.

Lemma 5.6. *Let (e_1, e_2, e_3) be an orthonormal basis for M_p such that $e_1 = \sigma_p'(0)$ and $e_2 = \gamma_1'(0)$. Then for any $X \in C_p$ given by*

$$X/\|X\| = e_1 \cos\alpha + e_2 \sin\alpha \cos\beta + e_3 \sin\alpha \sin\beta \qquad (0 \leq \alpha \leq 2\pi, 0\, \beta \leq 2\pi)\,,$$

we have $\|X\| = \cot^{-1}(\cos\alpha \cot \pi/k)$. *Let* $X_1 \in C_p$ *be defined by* $\exp_p X_1 = \exp_p X \in C(p)$, *where X is given by the above equation and* $\alpha \neq \pi/2$. *Then we have*

$$X_1 = \cot^{-1}(\cos\alpha \cot \pi/k)[e_1 \cos(\pi - \alpha) + e_2 \sin(\pi - \alpha)\cos(\beta + 2\pi/k) + e_2 \sin(\pi - \alpha)\cos(\beta + 2\pi/k)]\,.$$

Hence the identifying structure of C_p under $\exp_p$ *is completely known.*

Proof. Since $d(p, \sigma_q(t)) = \pi/2$ holds for all $t \in [0, 2\pi]$, there exist t_0 and the compact totally geodesic hypersurface $\mathscr{F}^{q(t_0),p}$ a sheet of whose tangent planes at p is spanned by e_1 and $e_2 \cos\beta + e_3 \sin\beta$. Then we find $t_0 = \beta$, and also see that $\mathscr{F}^{\sigma(\beta),p}$ is obtained by $\pi(S^2(\tilde{\sigma}(\beta)))$. There is a geodesic triangle on $S^2(\tilde{\sigma}_{\tilde{q}}(\beta))$ with vertices $\exp_{\tilde{p}} \tilde{X}$, $\tilde{p}$ and $\tilde{\sigma}_{\tilde{p}}(2\pi/k)$ satisfying $\sphericalangle(\exp_{\tilde{p}} \tilde{X}, \tilde{p}, \tilde{\sigma}_{\tilde{p}}(2\pi/k)) = \sphericalangle(\exp_{\tilde{p}} \tilde{X}, \tilde{\sigma}_{\tilde{p}}(2\pi/k), \tilde{p}) = \alpha$, where we define $d\pi(\tilde{X}) = X$, $\tilde{X} \in \tilde{M}_{\tilde{p}}$. Then the cosine rule of spherical trigonometry implies that $\|X\| = \cot^{-1}(\cos\alpha \cot \pi/k)$. It is easily seen that $\sphericalangle(X_1, \sigma_p'(0)) = \pi - \sphericalangle(\exp_{\tilde{p}} \tilde{X}, \tilde{\sigma}_{\tilde{p}}(2\pi/k), \tilde{p}) = \pi - \alpha$ because π is a local isometry.

Remark. As for a vector $X = (\pi/2)(e_2 \cos\beta + e_3 \sin\beta)$, putting $X_i = (\pi/2)\{e_2 \cos(\beta + 2\pi i/k) + e_3 \sin(\beta + 2\pi i/k)\}$, $i = 1, 2, \cdots, k$ we have $\exp_p X_i = \exp_p X$.

As the final step, we shall study the tangent cut locus $C_{p^*}^*$ of the lens space $M^* = L(1, k)$. The universal covering manifold of M^* is S_1^3. Let $g \in G$ be the generator of the cyclic group G of order k, where k is an odd prime.

For arbitrary point $\tilde{x} \in S_1^3$, we have $\sum_{i=1}^{k} g^i(\tilde{x}) = 0$, from which the points $g(\tilde{x})$,

$\cdots, g^k(\tilde{x}) = \tilde{x}$ lie on a great circle of S_1^3 and divide the great circle into equal parts of length $2\pi k$. Putting $x^* = \pi(\tilde{x})$, there is a closed geodesic in M^* with length $2\pi/k$ which starts at x^* and is obtained from the image of the great circle containing $g^i(x)$ under π. We also see that Max $\{d(x^*, y^*) \,|\, y^* \in M\} = \pi/2$.

Let (u, v, w) be a local coordinate system of S_1^3 defined by

$$x(u, v, w) = \cos u \cos v \cdot E_1 + \sin u \cos v \cdot E_2 + \cos w \sin v \cdot E_3 + \sin w \sin v \cdot E_4 \,,$$

where (E_1, E_2, E_3, E_4) is the orthonormal basis for R^4. A totally geodesic hypersurface $S^2(\tilde{q})$ is expressed locally by $w = w_0$ which is a two-sphere in S_1^3 with the north pole $\tilde{q}$ given by $q = \cos w_0 \cdot E_3 + \sin w_0 \cdot E_4$ and the equator given by $u \longrightarrow \cos u \cdot E_1 + \sin u \cdot E_2$. Since $S^2(\tilde{q})$ is of constant curvature 1 and π is a local isometry, $\pi(S^2(\tilde{q}))$ is also compact and of constant curvature 1 with self intersection in such a way that the image of equator is a closed geodesic of length $2\pi/k$ and is covered k times by the equator $u \longrightarrow \cos u \cdot E_1 + \sin u \cdot E_2$. We see that any other point on $\pi(S^2(q))$ has no intersection.

Let $\tilde{\Sigma}_{\tilde{p}} = \{\tilde{\sigma}_{\tilde{p}}(u)\}$ $(0 \leq u \leq 2\pi)$ be defined by $\tilde{\sigma}_{\tilde{p}}(u) = \cos u \cdot E_1 + \sin u \cdot E_2$ where we put $\tilde{p} = (1, 0, 0, 0)$ or $\tilde{p}(u, v, w) = (0, 0, 0)$, and $\pi(\tilde{p}) = p^*$. We see that the cut locus of $\pi(S^2(\tilde{q}))$ with respect to $p^* \in \pi(S^2(\tilde{q}))$ is contained entirely in the cut locus $C^*(p^*)$ of M^*. Putting $\tilde{\Lambda}_u = \{\tilde{\lambda}_u(v)\}$ $(0 \leq v \leq \pi/2)$, $\tilde{\lambda}_u(0) = \tilde{\sigma}_{\tilde{p}}(u)$ and $\tilde{\lambda}_u(\pi/2) = \tilde{q}$, $\pi \circ \tilde{\Sigma}_{\tilde{p}} = \Sigma^*_{p^*}$, $\sigma^*_{p^*}(0) = p^*$ and $\pi \circ \tilde{\Lambda}_u = \Lambda^*_u$, $\lambda^*_u(0) = \sigma^*_{p^*}(u)$, the cut locus of $\pi(S^2(\tilde{q}))$ with respect to $p^* = \sigma^*_{p^*}(0)$ is the set $\{\Lambda^*_u \,|\, [-\pi/2, \pi/2] \,|\, u = (2i-1)\pi/k, i = 1, 2, \cdots, k\}$. Denoting by $\tilde{\Gamma}_i$ the geodesic in S_1^3 joining $\tilde{\sigma}_{\tilde{p}}(2\pi i/k)$ to $\tilde{q}$, i.e., $\tilde{\Gamma}_i = \tilde{\Lambda}_{2\pi i/k}$, we see the angle between $g^j \circ \tilde{\Gamma}_i$ and $g^{j+1} \circ \tilde{\Gamma}_{i-1}$ at $\tilde{p}$ is equal to $2\pi/k$ for every $j, i = 1, 2, \cdots, k \pmod{k}$. This fact shows that the angle between Γ^*_i and Γ^*_{i+1} at p^* is equal to $2\pi/k$ for $i = 1, \cdots, k$. We also see that the angle between $\Sigma^*_{p^*}$ and Γ^*_i is equal to $\pi/2$.

Denoting $\mathscr{F}^{q^*, p^*} = \pi(S^2(\tilde{q}))$, where $q^* = \pi(\tilde{q})$, we have the same arguments for the tangent space $(\mathscr{F}^{q^*, p^*})_{p^*}$ at p^* as those of $\mathscr{F}^{q, p}$, and the family $\{\mathscr{F}^{\sigma^* q^*(t), p^*}\}$ $(0 \leq t \leq 2\pi)$ generates M^*. Then we have the same argument as that in Lemma 5.6 for $C^*_{p^*} \subset M^*_{p^*}$.

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